Solution 4

1. Let $C^k[a,b]$ be the vector space of all k-th continuously differentiable functions on $[a,b]$. Show that $\|f\|_k \equiv \sum_{j=0}^{k} \|f^{(j)}\|_\infty$ defines a norm on $C^k[a,b]$. Furthermore, $f_n \to f$ in $(C[a,b], \|\cdot\|_k)$ means $f_n \rightrightarrows f, \cdots, f_n^{(k)} \rightrightarrows f^{(k)}$.

   Solution. Straightforward.

2. Let $C^\infty[a,b]$ be the vector space of all infinitely many times differentiable functions on $[a,b]$. Show that

$$d_\infty(f,g) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{\|f-g\|_k}{1 + \|f-g\|_k}$$

   defines a metric on $C^\infty[a,b]$ such that $f_n \to f$ means $\|f_n - f\|_k \to 0$ for all k.

   Solution. Use the fact that $\rho = d/(1+d)$ is metric bounded by 1.

3. In class we showed that the set $P = \{f : f(x) > 0, \forall x \in [a,b]\}$ is an open set in $C[a,b]$. Show that it is no longer true if the norm is replaced by the L^1-norm. In other words, for each $f \in P$ and each $\varepsilon > 0$, there is some continuous g which is negative somewhere such that $\|g - f\|_1 < \varepsilon$.

   Solution. Fix a point, say, a and consider the continuous piecewise function φ_k which is equal to 1 at a and vanishes on $[a + 1/k, b]$. Then

$$\int_a^b \varphi_k(x)dx = \frac{1}{2k} \ .$$

   Let $f \in C[a,b]$ and $g_k = f - (f(a)+1)\varphi_k$ also belongs to $C[a,b]$ and $g_k(a) = -1 < 0$, but

$$\|f - g_k\|_1 = \int_a^b |f(x) - g_k(x)|dx = \frac{f(a)+1}{2k} \to 0$$

   as $k \to \infty$.

4. Show that $[a,b]$ can be expressed as the intersection of countable open intervals. It shows in particular that countable intersection of open sets may not be open.

   Solution. Simply observe

$$[a,b] = \bigcap_{j=1}^{\infty} (a - 1/j, b + 1/j) \ .$$

5. Optional. Show that every open set in $\mathbb{R}$ can be written as a countable union of disjoint open intervals. Suggestion: Introduce an equivalence relation $x \sim y$ if x and y belongs to the same open interval in the open set and observe that there are at most countable many such intervals.

   Solution.

   Let V be open in $\mathbb{R}$. Fix $x \in V$, there exists some open interval I, $x \in I$, $I \subseteq V$. Let $I_\alpha = (a_\alpha, b_\alpha)$, $\alpha \in \mathcal{A}$, be all intervals with this property. Let

$$I_x = (a_x, b_x), a_x = \inf_\alpha a_\alpha, b_x = \sup_\alpha b_\alpha.$$

satisfy $x \in I_x$, $I_x \subseteq V$ (the largest open interval in V containing x). It is obvious that $I_x \cap I_y \neq \phi \Rightarrow I_x = I_y$. Let $x \sim y$ if $I_x = I_y$. Then one can show that $\sim$ is an equivalence relation. By the discussion above, we have

$$V = \bigcup_{x \in V} I_x = \bigcup_{[x] \in V/\sim} \left(\bigcup_{y \sim x} I_x \right) = \bigcup_{[x] \in V/\sim} I_x,$$

which is a disjoint union. Moreover $V/\sim$ is at most countable since we can pick a rational number in each I_x to represent the class $[x] \in V/\sim$. Thus V can be written as a countable union of disjoint open intervals.

6. Identify the boundary points, interior points, interior and closure of the following sets in $\mathbb{R}$:

    (a) $[1,2) \cup (2,5) \cup \{10\}$.

    (b) $[0,1] \cap \mathbb{Q}$.

    (c) $\bigcup_{k=1}^{\infty} (1/(k+1), 1/k)$.

    (d) $\{1, 2, 3, \cdots\}$.

    Solution.

    (a) Boundary points: $1, 2, 5, 10$. Interior points: $(1,2), (2,5)$. Interior: $(1,2) \cup (2,5)$. Closure: $[1,5] \cup \{10\}$.

    (b) Boundary points: All points in $[0,1]$. No interior point. Interior: the empty set ϕ. Closure: $[0,1]$

    (c) Boundary points: $\{1/k : k \geq 1\}, 0$. Interior points: all points in this set. Interior: This set (because it is an open set). Closure: $[0,1]$.

    (d) Boundary points $1, 2, 3, \cdots$. No interior points. Interior: ϕ. Closure: the set itself (it is a closed set).

7. Identify the boundary points, interior points, interior and closure of the following sets in $\mathbb{R}^2$:

    (a) $R \equiv [0,1) \times [2,3) \cup \{0\} \times (3,5)$.

    (b) $\{(x,y) : 1 < x^2 + y^2 \leq 9\}$.

    (c) $\mathbb{R}^2 \setminus \{(1,0), (1/2,0), (1/3,0), (1/4,0), \cdots\}$.

    Solution.

    (a) Boundary points: the geometric boundary of the rectangle and the segment $\{0\} \times [3,5]$. Interior points: all points inside the rectangle. Interior $(0,1) \times (3,5)$. Closure $[0,1] \times [3,5] \cup \{0\} \times [3,5]$.

    (b) Boundary points: all (x,y) satisfying $x^2 + y^2 = 1$ or $x^2 + y^2 = 9$. Interior points: all points satisfying $1 < x^2 + y^2 < 9$. Interior $\{(x,y) : 1 < x^2 + y^2 < 9\}$. Closure $\{(x,y) : 1 \leq x^2 + y^2 \leq 9\}$.

    (c) Boundary points: $(0,0), (1,0), (1/2), (1/3,0), \cdots$. Interior points: all points except boundary points. Interior: $\mathbb{R}^2 \setminus \{(0,0), (1,0), (1/2), (1/3,0), \cdots\}$. Closure: $\mathbb{R}^2$.

8. Describe the closure and interior of the following sets in $C[0,1]$:

(a) $\{f : f(x) > -1, \; \forall x \in [0,1]\}$.

(b) $\{f : f(0) = f(1)\}$.

Solution.

(a) Closure: $\{f \in C[0,1] : f(x) \geq -1, \; \forall x \in [0,1]\}$. Interior: The set itself. It is an open set.

(b) Closure: The set itself. It is a closed set. Interior: ϕ. Let f satisfy $f(0) = f(1)$. For every $\varepsilon > 0$, it is clear we can find some $g \in C[0,1]$ satisfying $\|g - f\|_\infty < \varepsilon$ but $g(0) \neq g(1)$. It shows that every metric ball $B_\varepsilon(f)$ must contain some functions lying outside this set.

9. Find subsets in $\mathbb{R}$ such that $\overline{A \cap B}$ is properly contained in $\overline{A} \cap \overline{B}$.

   Solution.

   $\overline{A \cap B} = \overline{A} \cap \overline{B}$ is not always true. For instance, consider intervals (a,b) and (b,c). We have $\overline{(a,b)} \cap \overline{(b,c)} = \{b\}$ but $\overline{(a,b) \cap (b,c)} = \phi$. Or you take A to be the set of all rationals and B all irrationals. Then $\overline{A \cap B} = \overline{\phi} = \phi$ but $\overline{A} \cap \overline{B} = \mathbb{R}$!

10. Show that $\overline{E} = \{x \in X : d(x, E) = 0\}$ for every non-empty $E \subset X$.

    Solution. Let $A = \{x \in X : d(x, E) = 0\}$. Claim that A is closed. Let $x_n \to x$ where $x_n \in A$. Recalling that $x \mapsto d(x, E)$ is continuous, so $d(x, E) = \lim_{n \to \infty} d(x_n, E) = 0$, that is, $x \in A$. We conclude that A is a closed set. As it clearly contains E, so $\overline{E} \subset A$ since the closure of E is the smallest closed set containing E. On the other hand, if $x \in A$, then $B_{1/n}(x) \cap E \neq \phi$. Picking $x_n \in B_{1/n}(x) \cap E$, we have $\{x_n\} \subset E$, $x_n \to x$, so $x \in \overline{E}$.